Geometry of Uncertainty Relations for Linear Combinations of Position and Momentum

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Abstract

For a quantum particle with a single degree of freedom, we derive preparational sum and product uncertainty relations satisfied by N linear combinations of position and momentum observables. The bounds depend on their *degree of incompatibility* defined by the area of a parallelogram in an N-dimensional coefficient space. Maximal incompatibility occurs if the observables give rise to regular polygons in phase space. We also conjecture a Hirschman-type uncertainty relation for N observables linear in position and momentum, generalizing the original relation which lower-bounds the sum of the position and momentum Shannon entropies of the particle.

1 Introduction

For a long time, quantum mechanical uncertainty relations were tantamount to statements about *pairs* of non-commuting observables. Heisenberg's discussion of a fictitious γ-ray microscope in 1927 [1] led Kennard to immediately derive a rigorous *preparational* uncertainty relation [2] for the product of the variances of position and momentum observables. The existence of pairwise incompatible observables represents one of the defining features of quantum theory.

It is natural to suspect that similar limitations may also exist for triples, quadruples, etc. of non-commuting observables, and they may not be reducible to uncertainty relations for pairs. Indeed, the triple uncertainty relation [3], for example,

$$\Delta p \Delta q \Delta r \geq \left(\tau \frac{\hbar}{2}\right)^{3/2}, \quad \tau = \csc\left(\frac{2\pi}{3}\right) \simeq 1.15\,, \tag{1}$$

bounds the product of the variances of *three pairwise canonical* operators, $\hat{p}$, $\hat{q}$, and $\hat{r} = -\hat{p} - \hat{q}$. The bound (1) follows neither from individually applying Heisenberg's uncertainty relation

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to each of the canonical pairs of observables $(\hat{p},\hat{q})$, $(\hat{q},\hat{r})$ and $(\hat{r},\hat{p})$, nor from its generalization found by Robertson and Schrödinger [4, 5]. Early on, Robertson derived inequalities for sets of more than two observables [6] but the results do not cover the situation we will consider. For example, his bound on the product of the variances of the observables $\hat{p},\hat{q}$, and $\hat{r}$ turns out to be the trivial one, $\Delta p \Delta q \Delta r \geq 0$.

For a long time, uncertainty relations for continuous variables were thought to be of mainly conceptual interest. For systems with more than one degree of freedom, however, they are now known to provide tools to detect entanglement. The criteria may, for example, either use the variances of position and momentum operators only, as in [7], or the entire covariance matrix [8]. Not surprisingly, the triple uncertainty relation (1) also lends itself to detect entanglement, according to a recent proposal and its quantum optical realization [9].

In this paper, we will derive tight inequalities for the product and the sum of variances of finitely many observables for a single continuous variable describing, for example, a quantum particle restricted to move on the real line. We limit ourselves to *linear combinations* of position and momentum observables. Recent work on uncertainty relations beyond pairs of observables [10, 11] has led to new *state-dependent* bounds, as well as to bounds on the variances of multiple *unitary* operators [12]. In contrast to these approaches, the *linearity* of the observables we consider will lead to *state-independent* bounds, for the traditional case of Hermitean observables.

We will also introduce a many-observable generalization of the *entropic* uncertainty relation conjectured by Hirschman [13] in 1957 (but proved only two decades later [14, 15]). It is, in fact, straightforward to ask for a bound on the sum of *more than two* Shannon entropies for a given quantum state. As for variance-based uncertainty relations, we again expect Gaussian states to play an important role, suggested by the fact that they *saturate* the proposed inequalities. Recent results for entropic uncertainty relations valid in Hilbert spaces of small finite dimensions show how difficult it is to obtain tight bounds [16].

We have laid out this paper in the following way. In Sec. 2 we derive inequalities obeyed by the variances of N observables linear in position and momentum. The "non-commutativity" encoded in their pairwise commutators can be expressed in the *degree of incompatibility*, i.e. a real number which determines the lower bounds on sums and products of variances. Geometrically, this degree is given by the area of a suitably defined parallelogram in coefficient space $\mathbb{R}^N$. Specific sets of observables associated with regular polygons are shown to saturate the bounds. In Sec. 3 we generalize Hirschman's entropic uncertainty relation to more than two observables and explain why we expect the conjectured bounds to be tight. The last section summarizes our results and we will draw conclusions.

2 Variance-based uncertainty relations

The position and momentum observables $\hat{p}$ and $\hat{q}$ of a single quantum particle act on Hilbert space the elements of which can be represented as square-integrable functions over the real line, i.e. $\mathscr{H} = L^2(\mathbb{R})$. We introduce N Hermitean operators by combining them linearly,

$$\hat{r}_j = a_j \hat{p} + b_j \hat{q}\,, \qquad a_j, b_j \in \mathbb{R}\,, \qquad j = 1 \ldots N\,. \tag{2}$$

To exclude a trivial situation, at least two of the operators $\hat{r}_j, j = 1 \ldots N$, should not commute. Using a system of units in which both position and momentum have physical dimension $\sqrt{\hbar}$, the coefficients a_j, b_j, $j = 1 \ldots N$, are dimensionless. The operators in (2) represent *observables* since they can be measured as quadratures of an electromagnetic field in quantum-optical experiments, for example [17, 18, 19]. Each observable $\hat{r}_j$ is characterized by a vector in a two-dimensional Euclidean space,

$$\mathbf{r}_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix} \in \mathbb{R}^2 , \qquad j = 1 \dots N . \tag{3}$$

We will call $r_j = \sqrt{a_j^2 + b_j^2}$ the “length” of the observable $\hat{r}_j$.

The fundamental commutation relation

$$[\hat{p}, \hat{q}] = \frac{\hbar}{i} \hat{I} \tag{4}$$

where $\hat{I}$ is the the identity operator on the Hilbert space $\mathscr{H}$, implies that the pairwise commutators of the r-observables are given by

$$\left[\hat{r}_j, \hat{r}_k\right] = A(\mathbf{r}_j, \mathbf{r}_k) \frac{\hbar}{i} \hat{I} , \qquad j,k = 1 \dots N , \tag{5}$$

with the function $A(\cdot,\cdot)$ computing the (signed) area of the parallelogram determined by the vectors $\mathbf{r}_j, \mathbf{r}_k \in \mathbb{R}^2$,

$$A(\mathbf{r}_j, \mathbf{r}_k) = \left(a_j b_k - a_k b_j\right) \equiv A_{jk} . \tag{6}$$

Thus, the commutation relations between the r-observables are encoded in the N-by-N skew-symmetric matrix $\mathbf{A}$ with matrix elements $A_{jk} = -A_{kj}$. This antisymmetric structure finds its natural expression in a coordinate-independent formulation. Let us treat the linear combinations $\hat{r}_j$, $j = 1 \dots N$, as components of a vector operator with N components,

$$\hat{\mathbf{r}} = \begin{pmatrix} \hat{r}_1 \\ \vdots \\ \hat{r}_N \end{pmatrix} \equiv \mathbf{a}\hat{p} + \mathbf{b}\hat{q} , \qquad \mathbf{a}, \mathbf{b} \in \mathbb{R}^N . \tag{7}$$

Since the components of the exterior product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, are given by

$$(\mathbf{u} \wedge \mathbf{v})_{jk} = u_j v_k - u_k v_j , \quad j,k = 1 \dots N , \tag{8}$$

we find that the N^2 commutation relations (5) elegantly combine to

$$\hat{\mathbf{r}} \wedge \hat{\mathbf{r}} = \mathbf{a} \wedge \mathbf{b} \frac{\hbar}{i} \hat{I} . \tag{9}$$

Normally, the wedge product of a vector with itself is equal to zero but this does not apply to the left-hand-side of (9) because $\hat{\mathbf{r}}$ is a vector with operator-valued, *non-commuting* components. The relation is consistent with writing $\hat{\mathbf{r}} = \sum_{j=1}^N \hat{r}_j \mathbf{e}_j$, where the vectors $\mathbf{e}_j$, $j = 1 \dots N$, form the standard orthonormal basis of the space $\mathbb{R}^N$, and using the anti-symmetry of the exterior products $\mathbf{e}_j \wedge \mathbf{e}_k = -\mathbf{e}_k \wedge \mathbf{e}_j$. In a similar spirit, the commutation relations for a spin, $\sum_{qr} \varepsilon_{pqr} \hat{s}_q \hat{s}_r = i\hat{s}_p$, $p, q, r \in \{x, y, z\}$, can be written formally as a cross product, $\hat{\mathbf{s}} \times \hat{\mathbf{s}} = i\hbar\hat{\mathbf{s}}$, by combining the three operator-valued components of a quantum spin in a single vector $\hat{\mathbf{s}} = \frac{\hbar}{2}\hat{\sigma}$ (see [20], for example). It will be useful to write Eq. (8) in vector form, i.e. $\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}$, where the outer product $\mathbf{u} \otimes \mathbf{v}$ of two vectors is defined by

$$(\mathbf{u} \otimes \mathbf{v})_{jk} \equiv \left(\mathbf{u}\mathbf{v}^T\right)_{jk} , \qquad j,k = 1 \dots N . \tag{10}$$

The squared norm or *magnitude* of the bi-vector $\mathbf{a} \wedge \mathbf{b} \in \bigwedge^2(\mathbb{R})$ is given by

$$|\mathbf{a} \wedge \mathbf{b}|^2 = \sum_{j>k=1}^{N} A_{jk}^2 \,. \tag{11}$$

It has a simple expression in terms of the vectors defining the r-operators,

$$|\mathbf{a} \wedge \mathbf{b}|^2 = \sum_{j>k=1}^{N} \left(a_j b_k - a_k b_j\right)^2 = |\mathbf{a}|^2 \, |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \,, \tag{12}$$

which follows from Lagrange's identity for real numbers. Using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\phi$, where $\phi \in [0, \pi)$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, one finds

$$|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\phi \,, \tag{13}$$

in agreement with the wedge product being a generalization of the vector product in $\mathbb{R}^3$.

Geometrically, the squared norm of a bi-vector $\mathbf{a} \wedge \mathbf{b}$ is given by the sum of the squared areas of the parallelograms defined by all pairs of vectors $\mathbf{r}_j \in \mathbb{R}^2$, $j = 1 \ldots N$, which, according to (12), equals the square of the area of the parallelogram spanned by the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ in coefficient space. Not surprisingly, the norm is also closely related to a norm of the antisymmetric matrix $\mathbf{A}$ defined by Eq. (5): the square of its Frobenius (or Hilbert-Schmidt or $L_{2,2}$) norm reads

$$\|\mathbf{A}\|_F^2 = \mathrm{Tr}\left(\mathbf{A}^T \mathbf{A}\right) = \sum_{j,k=1}^{N} A_{jk}^2 = 2 \sum_{j>k=1}^{N} A_{jk}^2 = 2\,|\mathbf{a} \wedge \mathbf{b}|^2 \,. \tag{14}$$

This relation will be used in Sec. 2.3

2.1 Sum and product inequalities

The variances $\Delta^2 r_j \equiv \langle\psi|\hat{r}_j^2|\psi\rangle - \langle\psi|\hat{r}_j|\psi\rangle^2$ of the N linearly dependent r-observables in a pure state $|\psi\rangle \in \mathscr{H}$ are given by

$$\Delta^2 r_j = a_j^2 \Delta^2 p + b_j^2 \Delta^2 q + 2 a_j b_j C_{pq} \,, \qquad j = 1 \ldots N \,, \tag{15}$$

where we have introduced the covariance

$$C_{pq} = \frac{1}{2}\left(\langle\psi|\left(\hat{p}\hat{q} + \hat{q}\hat{p}\right)|\psi\rangle - \langle\psi|\hat{p}|\psi\rangle\langle\psi|\hat{q}|\psi\rangle\right) \,. \tag{16}$$

Adding the variances $\Delta^2 r_j$, we obtain

$$\sum_{j=1}^{N} \Delta^2 r_j = |\mathbf{a}|^2 \Delta^2 p + |\mathbf{b}|^2 \Delta^2 q + 2\,\mathbf{a} \cdot \mathbf{b}\, C_{pq} \,. \tag{17}$$

The right-hand-side of Eq. (17) defines a functional of three operators quadratic in position and momentum. The bounds of such expressions have been studied systematically in [21]. An explicit, non-trivial lower bound has been obtained for the linear combination of the variances $\Delta^2 p, \Delta^2 q$ and the covariance C_{pq} (see Eq. (72) of [21]),

$$\mu \Delta^2 p + \nu \Delta^2 q + 2\lambda C_{pq} \geq \hbar \sqrt{\mu\nu - \lambda^2}, \qquad \mu, \nu > 0 \,, \quad \mu\nu > \lambda^2 \,. \tag{18}$$

This inequality follows directly and elegantly from the non-negative expectation value of a quadratic form in position and momentum in an arbitrary quantum state $\hat{\rho}$,

$$\mathrm{Tr}\left[\hat{z}\,\hat{\rho}\,\hat{z}^{\dagger}\right]=\mathrm{Tr}\left[\left(\hat{z}\,\hat{\rho}^{1/2}\right)\left(\hat{z}\,\hat{\rho}^{1/2}\right)^{\dagger}\right]\geq 0\,, \tag{19}$$

where

$$\hat{z}=\alpha\left(\hat{p}-\langle\hat{p}\rangle\right)+\beta\left(\hat{q}-\langle\hat{q}\rangle\right)\,,\quad \langle\hat{p}\rangle\equiv\mathrm{Tr}\left[\hat{\rho}\,\hat{p}\right]\,,\quad \text{etc.}\,, \tag{20}$$

and $\alpha, \beta \in \mathbb{R}$, are complex numbers which satisfy $\mathrm{Im}\left(\alpha\,\beta^*\right) \geq 0$. A straightforward calculation shows that upon identifying $\mu \equiv |\alpha|^2\,, \nu = |\beta|^2$ and $\lambda = \mathrm{Re}\left(\alpha^*\,\beta\right)$ one obtains indeed (18), valid for both pure *and* mixed states.

Setting

$$\mu\equiv|\mathbf{a}|^2\,,\qquad \nu\equiv|\mathbf{b}|^2\,,\qquad \lambda\equiv\mathbf{a}\cdot\mathbf{b}\,, \tag{21}$$

we can apply the tight inequality (18) since $|\mathbf{a}|^2, |\mathbf{b}|^2 > 0$ and $|\mathbf{a}|^2\,|\mathbf{b}|^2 > (\mathbf{a}\cdot\mathbf{b})^2$ hold. Recalling the identity (12) leads to the *sum inequality* for arbitrary quantum states,

$$\sum_{j=1}^{N}\Delta^2 r_j \geq \hbar\,|\mathbf{a}\wedge\mathbf{b}|\,, \tag{22}$$

which is our first main result. Appendix A presents an alternative derivation which is based on the validity of (18) for pure states and the concavity of the variance.

Eq. (22) correctly reproduces both the pair and triple sum identities leading to the bounds $\hbar$ and $\hbar\sqrt{3}$, respectively. The bound is *state-independent* because the commutator between any two linear combinations of position and momentum is a scalar multiple of the identity. A trivial bound (zero) is obtained if the inequality (18) is applied to each term of the sum (17) separately, i.e. *before* instead of *after* the summation in (17).

Using (5) it is possible to express the lower bound of the inequality (22) in terms of the pairwise commutators between the N operators,

$$\left(\sum_{j=1}^{N}\Delta^2 r_j\right)^2 \geq \sum_{j>k=1}^{N}\left|\langle[\hat{r}_j,\hat{r}_k]\rangle\right|^2\,, \tag{23}$$

where the expectation values of the commutators are taken in the state $|\psi\rangle$. Thus, the sum of the variances of N different linear combinations $\hat{r}_j$ of position and momentum operators is seen to be bounded from below by the square root of the *sum of the modulus squared of all commutators between the operators*. Applying the Cauchy-Schwarz inequality to this expression for $N > 2$, we find that

$$\sum_{j>k=1}^{N}\left|\langle[\hat{r}_j,\hat{r}_k]\rangle\right|^2 > \left(\frac{1}{N-1}\sum_{j>k=1}^{N}\left|\langle[\hat{r}_j,\hat{r}_k]\rangle\right|\right)^2\,. \tag{24}$$

Upon concatenating this inequality with (23), we obtain a bound on the sum of N variances which can be derived directly from the inequalities valid for each of the $N(N-1)$ pairs $(\Delta^2 r_k + \Delta^2 r_j)$, $1 \leq k < j \leq N$. The stronger bound (22) shows that these uncertainty relations for N observables do *not* follow from those of the pairwise inequalities. According to [22], the concatenated inequality is actually known to hold for *arbitrary* observables $\hat{r}_j$, $j = 1\ldots N$, not just linear combinations of position and momentum. However, it is also not tight as the case of *three* observables shows [23, 24].

To identify the states saturating the inequality (22) let us introduce the ground state $|0\rangle$ of a harmonic quantum oscillator with unit mass and frequency, and the family of coherent states $|\alpha\rangle = \hat{T}_\alpha|0\rangle$, where the unitary operator

$$\hat{T}_\alpha = \exp\left[i\left(p_0\hat{q} - q_0\hat{p}\right)/\hbar\right] , \qquad \alpha = \frac{1}{\sqrt{2\hbar}}\left(q_0 + ip_0\right) , \tag{25}$$

generates a position and momentum translation by amounts q_0 and p_0. As shown in [21], the inequality (18) and hence the sum inequality (22) attain their minimum if the oscillator resides in a suitably *squeezed* ground state $|0\rangle$,

$$|\mu,\nu,\lambda\rangle = \hat{G}_{\frac{\lambda}{\nu}}\hat{S}_{\frac{1}{2}\ln\left(\frac{\nu}{\sqrt{\mu\nu-\lambda^2}}\right)}|0\rangle , \tag{26}$$

or in any state obtained from rigidly displacing it, i.e. $\hat{T}_\alpha|\mu,\nu,\lambda\rangle$. Here, the unitary operator

$$\hat{G}_g = \exp\left[ig\hat{p}^2/2\hbar\right] , \qquad g \in \mathbb{R} , \tag{27}$$

generates a *momentum gauge* transformation while

$$\hat{S}_\gamma = \exp\left[i\gamma\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)/2\hbar\right] , \qquad \gamma \in \mathbb{R} , \tag{28}$$

squeezes a state along the coordinate axes of phase space. For $N = 2$, with observables $\hat{r}_1 = \hat{p}$ and $\hat{r}_2 = \hat{q}$, say, corresponding to $\mu = \nu = 1$ and $\lambda = 0$, we find $\hat{G}_0 = \hat{S}_0 = \hat{I}$. This result agrees with the well-known fact that the only states minimizing the sum $\Delta^2 p + \Delta^2 q$ are given by the ground state of a harmonic oscillator and its rigid displacements in phase space.

Next, we wish to generalize Heisenberg's uncertainty relation by deriving a bound on the value of the *product* of the variances for the observables $\hat{r}_j, j = 1 \ldots N$,

$$J\left[|\psi\rangle\right] = \prod_{j=1}^{N} \Delta^2 r_j , \tag{29}$$

where $N \geq 2$. Using the identities (15), the functional $J\left[|\psi\rangle\right]$, which associates a number to each state $|\psi\rangle$, turns into a polynomial of order N in the basic variances $\Delta^2 p, \Delta^2 q$, and the covariance C_{pq}. Its lower bound could be determined by applying the method described in [21]. However, in this highly symmetric case, another method turns out to be simpler which enables us to minimize the product J while respecting the constraint given by the sum inequality (22).

A function $J(\vec{x})$ has a minimum in the presence of an inequality $g(\vec{x}) \leq 0$ if the Karush-Kuhn-Tucker (KKT) conditions [25] are satisfied,

$$\frac{\partial J(\vec{x})}{\partial x_j} + \kappa\frac{\partial g(\vec{x})}{\partial x_j} = 0 , \qquad j = 1 \ldots N , \tag{30}$$

$$\kappa g(\vec{x}) = 0 , \tag{31}$$

where κ is a positive constant yet to be determined. Identifying the variables x_j with the variances $\Delta^2 r_j, j = 1 \ldots N$, the constraint (22) reads $g(\vec{x}) \equiv c - \sum_j x_j \leq 0$, with the positive number $c = \hbar\left|\mathbf{a} \wedge \mathbf{b}\right|$.

The unique solution of the KKT conditions (30) is easily found to be

$$x_1 = x_2 = \ldots = x_N = \frac{c}{N} , \tag{32}$$

which implies that the smallest value of the functional $J[|\psi\rangle]$ is given by $(c/N)^N$. In terms of the original variables, we finally obtain the *product inequality* for the variances of N linear combinations of position and momentum,

$$\prod_{j=1}^{N} \Delta^2 r_j \geq \left(\frac{\hbar\,|\mathbf{a} \wedge \mathbf{b}|}{N} \right)^N , \tag{33}$$

our second main result. The lower bounds for Heisenberg's uncertainty relation and for the triple product uncertainty relation (1) are reproduced correctly. The bounds are symmetric in all pairs of the N observables $\hat{r}_j, j = 1 \ldots N$, and they display a neat structure which involves the exterior product of the momentum and position coefficients in $\mathbb{R}^N$. The result (33) is genuinely different from Robertson's inequalities for N observables [6] since already in the case of $N = 3$ only a trivial bound results, $\Delta p \Delta q \Delta r \geq 0$. In addition, the derivation of inequality (33) also applies to mixed states, i.e. $\Delta^2 r_j = \mathrm{Tr}\left(\hat{\rho}\hat{r}_j^2\right) - \left(\mathrm{Tr}\left(\hat{\rho}\hat{r}_j\right)\right)^2, j = 1 \ldots N$.

2.2 Regular polygons

Let us now determine the bounds for N observables arranged in a symmetric way. We assume that the tips of the vectors $\mathbf{r}_j \in \mathbb{R}^2, j = 1 \ldots N$, are located on a circle of radius $R \in (0, \infty)$, and that they are distributed homogeneously. Explicitly, we have

$$\hat{r}_j = \left(R \cos\varphi_j\right) \hat{p} + \left(R \sin\varphi_j\right) \hat{q}\,, \qquad \varphi_j = \frac{2\pi(j-1)}{N}\,, \qquad j = 1, \ldots, N\,. \tag{34}$$

The tips of the vectors define a regular polygon with N vertices in the space $\mathbb{R}^2$, as illustrated in Fig. (1). We will always align the first observable with the momentum operator, i.e. $\hat{r}_1 = R\hat{p}$. This choice is not a restriction since the commutation relations do not change under rotations in $\mathbb{R}^2$ (cf. Appendix B).

From a *structural* point of view, the value of the constant R is not important as it only rescales all observables. One natural choice to fix this scale is to require that any two adjacent observables form a canonical pair,

$$[\hat{r}_j, \hat{r}_{j+1}] = \frac{\hbar}{i}\hat{I}\,, \qquad \hat{r}_{N+1} \equiv \hat{r}_1\,, \qquad j = 1 \ldots N\,. \tag{35}$$

These conditions are satisfied if the circumradius R of the polygon takes the value

$$R_N = \frac{1}{\sqrt{\sin\Delta_N}}\,, \qquad \Delta_N = \frac{2\pi}{N}\,. \tag{36}$$

In this case, the parallelograms defined by any two consecutive vectors $\mathbf{r}_j$ and $\mathbf{r}_{j+1}$, which enclose the angle $2\pi/N$, have unit area area, $A(\mathbf{r}_j, \mathbf{r}_{j+1}) = 1$. As the angles between neighbouring vectors decrease with larger values of N, the circumradius of the polygon must increase as $R_N \simeq \sqrt{N}$ in order to ensure (35).

Since the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ have components

$$a_j = R_N \cos\varphi_j\,, \qquad b_j = R_N \sin\varphi_j\,, \qquad j = 1 \ldots N\,, \tag{37}$$

we find that

$$|\mathbf{a} \wedge \mathbf{b}| = \frac{NR_N^2}{2} \equiv \frac{N}{2\sin\Delta_N}\,. \tag{38}$$

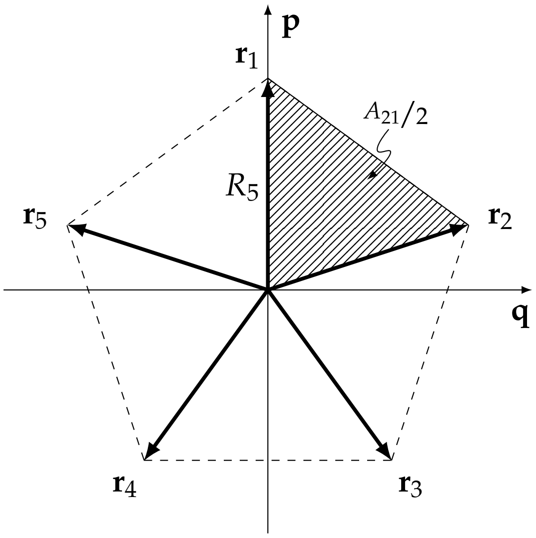

Figure 1: A regular pentagon in the dimensionless "phase space" $\mathbb{R}^2$, associated with the canonical operators $\hat{r}_j, j = 1 \ldots 5$, introduced in (34), of circumradius $R_5 = 1/\sqrt{\sin(2\pi/5)}$ (cf. Eq. (36)) and with area $A = 5/2$. The shaded triangle has half the size of the area $A_{21} \equiv A(\mathbf{r}_2, \mathbf{r}_1) \equiv 1$ given by the parallelogram spanned by the vectors $\mathbf{r}_2$ and $\mathbf{r}_1$ (cf. Eq. (6)).

Here we have used the trigonometric identities

$$\sum_{j=1}^{N} \sin^2\left(\frac{2\pi j}{N}\right) = \frac{N}{2} \qquad \text{and} \qquad \sum_{j=1}^{N} \sin\left(\frac{4\pi j}{N}\right) = 0\,, \tag{39}$$

to show that

$$|\mathbf{a}|^2 = |\mathbf{b}|^2 = \frac{NR_N^2}{2} \qquad \text{and} \qquad \mathbf{a}\cdot\mathbf{b} = 0\,, \tag{40}$$

respectively. Now the identity (12) implies that the sum and the product inequalities (see Eqs. (22) and (33)) for the variances of N observables associated with regular polygons are given by

$$\sum_{j=1}^{N} \Delta^2 r_j \geq \frac{N\hbar}{2\sin\Delta_N} \qquad \text{and} \qquad \prod_{j=1}^{N} \Delta^2 r_j \geq \left(\frac{\hbar}{2\sin\Delta_N}\right)^N\,, \tag{41}$$

respectively .

It is possible to absorb the factor $\sin\Delta_N$ on the right-hand-side of these inequalities by considering vectors $\mathbf{r}_j$ in (34) with tips located on the *unit* circle. In this case, the right-hand-side of the commutators (35) is found to be proportional to $\sin\Delta_N \simeq 1/\sqrt{N}$ since adjacent observables differ less and less for increasing values of N. Then, the bounds in Eqs. (41) take particularly simple forms,

$$\sum_{j=1}^{N} \Delta^2 r_j \geq N\frac{\hbar}{2} \qquad \text{and} \qquad \prod_{j=1}^{N} \Delta^2 r_j \geq \left(\frac{\hbar}{2}\right)^N\,, \tag{42}$$

i.e. each variance formally contributes at least an amount $\hbar/2$. The states that saturate these inequalities are the *coherent* states $|\alpha\rangle = \hat{T}_\alpha|0\rangle$, introduced via Eq. (25). If $N = 2$ or $N = 4$, the left-hand-side of the product inequality depends only on $\Delta p \Delta q$ which is invariant under squeezing transformations, hence leading to a *larger* family of extremal states, namely suitably squeezed states. Products of three (or more than four) variances do *not* exhibit this continuous symmetry.

2.3 Degrees of incompatibility

In this section, we will argue that the dependence of the sum and product bounds on only the norm $|\mathbf{a} \wedge \mathbf{b}|$ is not a coincidence. We will show that there exists a transformation which maps the vector operator $\hat{\mathbf{r}} = \mathbf{a}\hat{q} + \mathbf{b}\hat{p}$ to $\hat{\mathbf{r}}' = \mathbf{a}'\hat{q} + \mathbf{b}'\hat{p}$ in such a way that the commutation relations (9) assume their *standard form*,

$$\hat{\mathbf{r}}' \wedge \hat{\mathbf{r}}' = |\mathbf{a} \wedge \mathbf{b}|\; \mathbf{e}_1 \wedge \mathbf{e}_2 \frac{\hbar}{i} \hat{I}\,, \tag{43}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are a pair of orthogonal unit vectors in the coefficient space $\mathbb{R}^N$. Therefore, the commutation relations for N linear combination of position and momentum can be characterized by a single real number,

$$\mathrm{Inc}(\mathbf{a}, \mathbf{b}) \equiv |\mathbf{a} \wedge \mathbf{b}|\;, \tag{44}$$

measuring the *degree of incompatibility* of the observables $\hat{r}_j$, $j = 1 \ldots N$. The relation (43) states that the original commutation relations are equivalent to a situation in which all but two $\hat{r}$-observables have been mapped to 0,

$$\hat{r}_1' = |\mathbf{a} \wedge \mathbf{b}|^{1/2}\, \hat{p}\,, \qquad \hat{r}_2' = |\mathbf{a} \wedge \mathbf{b}|^{1/2}\, \hat{q}\,, \qquad \hat{r}_k' = 0\,, \quad k = 3 \ldots N\,, \tag{45}$$

corresponding to

$$\mathbf{a}' = |\mathbf{a} \wedge \mathbf{b}|^{1/2}\, \mathbf{e}_1 \qquad \text{and} \qquad \mathbf{b}' = |\mathbf{a} \wedge \mathbf{b}|^{1/2}\, \mathbf{e}_2\,, \tag{46}$$

respectively. We will obtain the standard form (43) by exploiting the fact that the norm of the bi-vector $\mathbf{a} \wedge \mathbf{b}$ is invariant under (i) gauge transformations and under (ii) transformations of the vector operator $\hat{\mathbf{r}}$ which are orthogonal in $\mathbb{R}^N$.

Before embarking on this calculation, we mention that other measures of incompatibility for *pairs* of observables exist. The *joint measurability region* [26] quantifies the incompatibility of two observables based on the amount of noise that needs to be added in order for them to become jointly measurable. Based on this notion a coarser measure can be introduced, the *joint measurability degree* [27], which returns a real number between 1/2 (corresponding to maximal incompatibility) and 1 (compatibility). For continuous variables, the pair of position and momentum is found to be maximally incompatible which agrees with the measure $\mathrm{Inc}(\mathbf{a}, \mathbf{b})$introduced here. However, the case of three or more observables has not been considered

To derive the relations (43), we first note that the observables

$$\hat{\mathbf{r}}_U = \hat{U}\, \hat{\mathbf{r}}\, \hat{U}^\dagger\,, \tag{47}$$

obtained from $\hat{\mathbf{r}}$ by any unitary operator $\hat{U}$, satisfy the same commutation relations as the original observables $\hat{\mathbf{r}}$. If we limit ourselves to *linear* canonical transformations of the observables $\hat{q}$ and $\hat{p}$, we find a set of transformations forming the group $Sp(2, \mathbb{R})$, generated by rotations, squeeze and gauge transformations described in [28].

Taking the unitary $\hat{U} = \hat{G}_g$ as defined in Eq. (27), position and momentum operators transform according to

$$\begin{aligned} \hat{p}_g &= \hat{p}\,, \\ \hat{q}_g &= \hat{q} + g\hat{p}\,, \qquad g \in \mathbb{R}\,. \end{aligned} \tag{48}$$

Clearly, the transformed coordinate vectors are $\mathbf{a}_g = \mathbf{a}$ and $\mathbf{b}_g = \mathbf{b} + g\mathbf{a}$. The components of the vector operator $\hat{\mathbf{r}}_g$ have the same commutators as those of $\hat{\mathbf{r}}$ as follows from the properties of the exterior product,

$$\mathbf{a}_g \wedge \mathbf{b}_g = \mathbf{a} \wedge (\mathbf{b} + g\mathbf{a}) = \mathbf{a} \wedge \mathbf{b} \,. \tag{49}$$

Geometrically, the parameter g labels a continuous family of parallelograms with sides $\mathbf{a}_g$ and $\mathbf{b}_g$. They all have the same area as they are related to each other by a shear transformation. If the parameter g takes the value

$$g_\perp = -\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \,, \tag{50}$$

the parallelogram turns into a *rectangle* spanned by two *orthogonal* vectors, $\mathbf{a}_\perp = \mathbf{a}$ and $\mathbf{b}_\perp = \mathbf{b} + g_\perp \mathbf{a}$.

The right-hand-side of the commutation relations $\hat{\mathbf{r}}_\perp \wedge \hat{\mathbf{r}}_\perp = i\hbar \, \mathbf{a}_\perp \wedge \mathbf{b}_\perp \, \hat{I}$ now depends on the orthogonal vectors $\mathbf{a}_\perp$ and $\mathbf{b}_\perp$. Denote unit vectors aligned with them by $\mathbf{e}_a$ and $\mathbf{e}_b$, respectively, and consider an orthogonal transformation $\mathbf{R}$, i.e. $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, which rotates the vector operator $\hat{\mathbf{r}}_\perp$ into

$$\hat{\mathbf{r}}' = \mathbf{R}\hat{\mathbf{r}}_\perp \,. \tag{51}$$

Note that, typically, such a transformation *cannot* be generated by a unitary operator acting on the fundamental pair $\hat{p}$ and $\hat{q}$. Since $\mathbf{e}_a \cdot \mathbf{e}_b = 0$, we can always find a transformation $\mathbf{R}$ which maps the vectors $\mathbf{e}_a$ and $\mathbf{e}_b$ to the first two elements of the standard basis,

$$\mathbf{e}_a = \mathbf{R}\mathbf{e}_1 \,, \qquad \mathbf{e}_b = \mathbf{R}\mathbf{e}_2 \,. \tag{52}$$

The rotation $\mathbf{R}$ is unique only for $N = 3$ since in $\mathbb{R}^3$ the map of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ determines the fate of the third basis vector, via $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Using the definition of the outer product in (10) and the fact that $\mathbf{R}^{-1} = \mathbf{R}^T$, one finds that

$$(\mathbf{R}\mathbf{a}) \otimes (\mathbf{R}\mathbf{b}) = (\mathbf{R}\mathbf{a})\,(\mathbf{R}\mathbf{b})^T = \mathbf{R}\left(\mathbf{a}\mathbf{b}^T\right)\mathbf{R}^T \,, \tag{53}$$

so that the exterior product (8) transforms according to

$$(\mathbf{R}\mathbf{a}) \wedge (\mathbf{R}\mathbf{b}) = \mathbf{R}\left(\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T\right)\mathbf{R}^T \equiv \mathbf{R}\,(\mathbf{a} \wedge \mathbf{b})\,\mathbf{R}^T \,. \tag{54}$$

The relation (14) now implies that the length of the bi-vector $\mathbf{a} \wedge \mathbf{b}$ is invariant under any rotation $\mathbf{R}$ applied to the N-component vector operator $\hat{\mathbf{r}}$,

$$|(\mathbf{R}\mathbf{a}) \wedge (\mathbf{R}\mathbf{b})|^2 = \frac{1}{2}\mathrm{Tr}\left[\left(\mathbf{R}\mathbf{A}\mathbf{R}^T\right)^T\left(\mathbf{R}\mathbf{A}\mathbf{R}^T\right)\right] = \frac{1}{2}\mathrm{Tr}\left(\mathbf{A}^T\,\mathbf{R}^T\mathbf{R}\,\mathbf{A}\,\mathbf{R}^T\mathbf{R}\right) = |\mathbf{a} \wedge \mathbf{b}|^2 \,. \tag{55}$$

Applying this property to the vector operator $\hat{\mathbf{r}}' = \mathbf{R}\,(\mathbf{a}_\perp \hat{p} + \mathbf{b}_\perp \hat{q})$, we finally obtain the desired result, Eq. (43). In general, the vectors $\mathbf{a}_\perp$ and $\mathbf{b}_\perp$ will be of different lengths. There is, however, a squeeze transformation which rescales the pair $\hat{q}, \hat{p}$, such that the lengths of the vectors will be equal (see Appendix B).

2.4 Maximal incompatibility

The degree of incompatibility $\mathrm{Inc}(\mathbf{a}, \mathbf{b})$ defined in Eq. (44) can take any non-negative value. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, the operators $\hat{r}_j$, $j = 1 \ldots N$, commute and hence are compatible, $\text{Inc}(\mathbf{a}, \lambda\mathbf{a}) = 0$, for all $\lambda \in \mathbb{R}$. Multiplying the operators $\hat{r}_j$ by a common factor $\lambda \in \mathbb{R}$, rescales their incompatibility accordingly,

$$\text{Inc}(\lambda\mathbf{a}, \lambda\mathbf{b}) = \lambda\, \text{Inc}(\mathbf{a}, \mathbf{b})\,. \tag{56}$$

To avoid artificially inflated values of incompatibility, it is natural to require that the vectors $\mathbf{r}_j$ which fix the operators $\hat{r}_j$, $j = 1 \ldots N$, have at most length one,

$$|\mathbf{r}_j|^2 \equiv r_j^2 \leq 1\,, \qquad j = 1 \ldots N\,. \tag{57}$$

This constraint is consistent with Heisenberg's uncertainty relation for the canonically conjugate pair of position and momentum observables.

It makes sense to ask for the *maximal* value which the incompatibility $\text{Inc}(\mathbf{a}, \mathbf{b}) = |\mathbf{a} \wedge \mathbf{b}|$ may take for N observables $\hat{\mathbf{r}}$. The maximum is of interest because it will determine the largest possible bounds for the sum and the product inequalities, by "exhausting" the quantum mechanical non-commutativity of the observables. Suppose we are given N observables defined by the vectors $\mathbf{r}_j = r_j \mathbf{u}_j$, $j = 1 \ldots N$, where each $\mathbf{u}_j$ is a unit vector and the lengths r_j satisfy (57). Then, the estimate

$$\text{Inc}^2(\mathbf{a}, \mathbf{b}) = \sum_{j>k=1}^{N} A^2(\mathbf{r}_j, \mathbf{r}_k) = \sum_{j>k=1}^{N} r_j^2 r_k^2 A^2(\mathbf{u}_j, \mathbf{u}_k) \leq \sum_{j>k=1}^{N} A^2(\mathbf{u}_j, \mathbf{u}_k) \tag{58}$$

shows that their incompatibility is smaller than that of N observables associated with the vectors $\widetilde{\mathbf{r}}_j = \mathbf{u}_j$, with all their tips located on the unit circle. Thus, maximal incompatibility will necessarily arise for an arrangement of N points on the unit circle.

It is instructive to discuss the simple case of $N = 2$. Position $\hat{q}$ and momentum $\hat{p}$ satisfy Heisenberg's uncertainty relation and should, of course, provide an example of maximal incompatibility. The incompatibility of any two observables with vectors $\mathbf{r}_j = r_j \mathbf{u}_j, j = 1, 2$, satisfying (57) and with $\mathbf{u}_1 \cdot \mathbf{u}_2 = \cos\phi$, is given by

$$\text{Inc}^2(\mathbf{a}, \mathbf{b}) \equiv (a_1 b_2 - a_2 b_1)^2 = r_1^2 r_2^2 \sin^2\phi \leq 1\,. \tag{59}$$

It achieves its maximum for $r_1 = r_2 = 1$ and $\phi = \pm\pi/2$. Thus, the pairs $(\hat{q}, \pm\hat{p})$ and all those obtained from rotating them by an angle $\theta \in [0, 2\pi)$ indeed max out the non-commutativity. The vectors $\mathbf{a}$ and $\mathbf{b}$ are necessarily orthogonal and of equal length. If the pair $(\mathbf{u}_1, \mathbf{u}_2)$ describes a configuration with maximal incompatibility, then all four configurations with vectors $(\pm\mathbf{u}_2, \pm\mathbf{u}_2)$ are also maximally incompatible. We ignore the uncertainty- preserving squeeze transformations here since they do not have an equivalent for other values of N.

Let us now search for the arrangements of not just two but N vectors with tips on the unit circle which will result in maximal incompatibility. Using the identity (12), we find

$$\text{Inc}(\mathbf{a}, \mathbf{b}) = |\mathbf{a}|\,|\mathbf{b}| \sin\phi\,, \qquad \phi \in [0, \pi)\,, \tag{60}$$

where the angle between the two vectors in $\mathbb{R}^N$ is defined by the relation $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\phi$. Summing the conditions $r_j^2 = a_j^2 + b_j^2 = 1$, $j = 1 \ldots N$, over all values of j, one finds $|\mathbf{b}|^2 = N - |\mathbf{a}|^2$ which implies

$$\text{Inc}(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \sqrt{N - |\mathbf{a}|^2}\, \sin\phi \leq |\mathbf{a}| \sqrt{N - |\mathbf{a}|^2} \leq \frac{N}{2}\,. \tag{61}$$

The last inequality follows because the function $f(x) = x\sqrt{N - x^2}$ has its unique maximum at $x = \sqrt{N/2}$. Thus, the incompatibility takes the value $N/2$ if there exist N observables

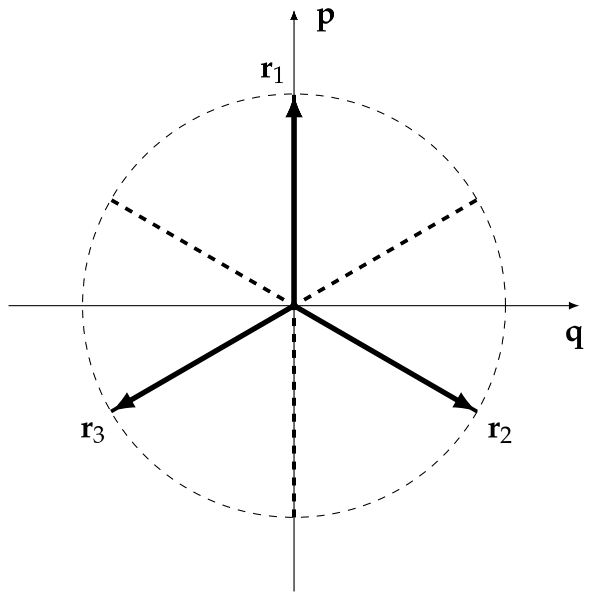

Figure 2: Phase-space visualization of three maximally incompatible observables: each of the eight triples $(\pm\mathbf{r}_1, \pm\mathbf{r}_2, \pm\mathbf{r}_3)$ corresponds to observables which maximise the incompatibility $\mathrm{Inc}(\mathbf{a}, \mathbf{b})$ since the variances $\Delta\hat{r}_j$ are invariant under $\hat{r}_j \to -\hat{r}_j$, $j = 1, 2, 3$. In addition, each configuration may be rotated rigidly by any angle between 0 and $2\pi/3$ without changing the value of the incompatibility. For more than three observables, the equilateral triangle with tips $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is replaced by a regular polygon with N vertices.

characterized by a pair $(\mathbf{a}, \mathbf{b})$ of vectors which are orthogonal and of equal length, $|\mathbf{a}| = |\mathbf{b}| = \sqrt{N/2}$.

According to Eq. (40), *regular polygons* with N vertices located on the unit circle ($R_N \equiv 1$) correspond precisely to this situation. Thus, we may conclude that the observables associated with the regular N-polygons introduced in Sec. 2.2 maximize the incompatibility inherent in N observables linear in position and momentum. Clearly, this set of observables is not the only one achieving the maximum: rotating of the polygon by any angle in the interval $(0, 2\pi/N)$ leads to equivalent arrangements, as do individual reflections of the vectors $\mathbf{r}_j$ about the origin.

We suspect that no other sets of N observables linear in position and momentum will lead to maximal incompatibility. However, we are only able to show this property for $N = 3$. Three observables as defined in (2) associated with unit vectors $\mathbf{r}_j$ are conveniently parameterized by

$$a_j = \cos\theta_j\,, \quad b_j = \sin\theta_j\,, \qquad \theta_j \in [0, 2\pi)\,, \qquad j = 1, 2, 3\,. \tag{62}$$

Their incompatibility is given by a function of two variables,

$$\begin{aligned}\mathrm{Inc}^2(\mathbf{a}, \mathbf{b}) &= \sum_{j>k=1}^{3} (a_j b_k - a_k b_j)^2 = \sum_{j>k=1}^{3} \sin^2\left(\theta_j - \theta_k\right) \\ &= \frac{3}{2} - \frac{1}{2}\sum_{j>k=1}^{3} \cos\left(2\left(\theta_j - \theta_k\right)\right)\,. \end{aligned} \tag{63}$$

Selecting the first observable to be momentum, $\hat{r}_1 = \hat{p}$, we have $\theta_1 = 0$. The maxima of the incompatibility occur when one of the angles θ_2 or θ_3 takes the value $\pi/3$ or $4\pi/3$ while the other becomes $2\pi/3$ or $5\pi/3$. The solutions for the observables $\hat{r}_2$ and $\hat{r}_3$ are shown in Fig. 2, in terms of the vectors $\mathbf{r}_j$ characterizing them. It is straightforward to confirm that the vectors **a** and **b** are indeed orthogonal for each set of observables maximizing the incompatibility.

3 Entropic uncertainty relations

Heisenberg's uncertainty relation expresses a fundamental restriction to simultaneously attribute specific values to both position and momentum of a quantum particle. Hirschman [13] used the position and momentum probability densities of a quantum state $|\psi\rangle$ to capture this feature without referring to variances of observables. Instead, he used the Shannon entropies of a state $|\psi\rangle$ associated with the modulus of the wave function in the position and momentum representation. Given the state $|\psi\rangle$ with position representation $\langle q|\psi\rangle = \psi(q)$, its *Shannon entropy*

$$S_q = -\int_{-\infty}^{\infty} dq\, |\psi(q)|^2 \log\left(\sqrt{\hbar}|\psi(q)|^2\right) , \tag{64}$$

returns small values for probability densities $|\psi(q)|^2$ which are localized and large ones for densities which are spread out; the factor $\sqrt{\hbar}$ ensures that the argument of the logarithm is dimensionless. The momentum representation of the state $|\psi\rangle$ follows from Fourier-transforming its position wave function,

$$\langle p|\psi\rangle = \psi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipq/\hbar}\psi(q)\, dq\,, \tag{65}$$

leading to a probability density $|\psi(p)|^2$ with Shannon entropy S_p, defined in analogy to Eq. (64). Hirschman showed that the sum of these entropies cannot fall below zero and conjectured that a tighter nonzero bound would hold,[1]

$$S_q + S_p \geq \ln(e\pi)\,. \tag{66}$$

Using the properties of a norm for the Fourier transform [30, 15], this uncertainty relation has been proved in [14], nearly 20 years after being conjectured.

The inequalities by Hirschman and Heisenberg are related closely. The variance of the observable $\hat{p}_\theta = \hat{p}\cos\phi + \hat{q}\sin\phi$, $\phi \in [0, 2\pi)$, has a lower bound [31]

$$\Delta^2 p_\phi \geq \frac{\hbar}{2e\pi} e^{2S_\phi}\,, \tag{67}$$

which depends on the Shannon entropy associated with the probability density $|\langle p_\phi|\psi\rangle|^2 = |\psi(p_\phi)|^2$, where $\hat{p}_\phi|p_\phi\rangle = p_\phi|p_\phi\rangle$ holds. Using Eq. (67) for both momentum and position (i.e. for $\phi = 0$ and $\phi = \pi/2$, respectively), the entropic inequality (66) indeed implies

$$\Delta^2 p\, \Delta^2 q \geq \left(\frac{\hbar}{2e\pi}\right)^2 e^{2(S_0 + S_{\pi/2})} \geq \left(\frac{\hbar}{2}\right)^2 , \tag{68}$$

as already pointed out by Hirschman [13]. If the system resides in the ground state of a harmonic oscillator with unit mass and frequency, i.e. in the coherent state $|0\rangle$, we have $\Delta^2 p = \Delta^2 q = \hbar/2$. Both inequalities in (68) are now saturated since Eq. (67) turns into an equality (which happens whenever the state is represented by a Gaussian [14]) so that $S_p = S_q = (1/2)\ln(e\pi)$. In other words, the value of the tight bound in Hirschman's

[1] This form of Hirschman's inequality holds if one sets a free dimensionless parameter equal to one as explained in Ref. [29].

inequality (66) is obtained if one considers a case in which the pair product-uncertainty relation is saturated and combines it with the bound (67).

This argument does not, of course, replace the proof of Hirschman's inequality. However, we use an analogous argument to conjecture a bound for a generalization of Hirschman's inequality which involves more than two observables linear in position and momentum. Consider $N \geq 3$ observables $\hat{r}_j = \hat{p}\cos\phi_j + \hat{q}\sin\phi_j$, $\phi_j = 2\pi(j-1)/N$, $j = 1 \ldots N$, associated with a regular N-polygon with vertices on the unit circle. The product inequality (42) is known to be saturated if the system resides in the state $|0\rangle$,

$$\prod_{j=1}^{N} \Delta^2 r_j = \left(\frac{\hbar}{2}\right)^N . \tag{69}$$

As the wave function of the state $|0\rangle$ is Gaussian in each $\hat{r}_j$-representation, we have

$$\Delta^2 r_j = \frac{\hbar}{2e\pi} e^{2S_j} , \qquad j = 1 \ldots N , \tag{70}$$

where S_j is the Shannon entropy of the probability density $|\psi(r_j)|^2$ of the state $|0\rangle$. Substituting (70) into (69), we find that

$$\frac{2}{N} \sum_{j=1}^{N} S_j = \ln(e\pi) \tag{71}$$

holds, leading to the conjecture of an N-observable Hirschman-type inequality,

$$\frac{2}{N}\left(S_1 + S_2 + \ldots + S_N\right) \geq \ln(e\pi) , \quad N \geq 3. \tag{72}$$

Other inequalities exist for the case of the r-observables defined by vertices distributed *inhomogeneously* on the unit circle since these configurations result in a smaller degree of incompatibility.

4 Summary and discussion

In this paper we have derived inequalities for N linear combinations of position and momentum of a quantum particle. The sum and the product inequality, Eqs. (22) and (33), depend on one single parameter only, the *degree of incompatibility* $\text{Inc}(\mathbf{a}, \mathbf{b})$ defined in Eq. (44). This number is the only remaining parameter once the original $N(N-1)$ commutator relations (9) have been brought to the standard form (43).

Using the relation between the arithmetic and the geometric mean, we can concatenate the two inequalities,

$$\frac{\Delta^2 r_1 + \Delta^2 r_2 + \ldots + \Delta^2 r_N}{N} \geq \left(\Delta^2 r_1 \, \Delta^2 r_2 \cdots \Delta^2 r_N\right)^{1/N} \geq \frac{\hbar\,|\mathbf{a} \wedge \mathbf{b}|}{N} , \tag{73}$$

neatly summarizing our main findings for the variances of multiple observables linear in position and momentum, valid for arbitrary (pure or mixed) quantum states. Given the product inequality, the bound of the geometric mean by the arithmetic mean actually provides an alternative derivation of the sum inequality. Heisenberg's inequality and the triple inequality emerge as the first two members of a family labeled by $N = 2, 3, \ldots$ The cases $N = 2$ and $N = 3$ are special since they are the only ones in which *all* pairwise commutators can be made to coincide.

Upon rescaling the observables by a common positive factor, $\hat{r}_j \to \hat{r}_j \sqrt{|\mathbf{a} \wedge \mathbf{b}|}$, $j = 1 \ldots N$, the inequalities (73) take a particularly simple form,

$$\frac{\Delta^2 r_1 + \Delta^2 r_2 + \ldots + \Delta^2 r_N}{N} \geq \left(\Delta^2 r_1 \, \Delta^2 r_2 \cdots \Delta^2 r_N \right)^{1/N} \geq \frac{\hbar}{N} , \tag{74}$$

showing immediately that saturation occurs if each variance takes the value $\hbar/N$. Only for $N = 2$ and $N = 4$, a one-parameter family of squeezed states (with *unequal* variances) exists which also saturate the second inequality (but not the first one).

To identify N linear observables $\hat{r}_j$ with *maximal* incompatibility, we have considered sets characterized by vectors $\mathbf{r}_j, j = 1 \ldots N$, of unit length or less. In this case, the bound on the right-hand-side of Eq. (73) has been found to reach its maximum if the N-dimensional coefficient vectors satisfy the condition $|\mathbf{a} \wedge \mathbf{b}| = N/2$. This happens, for example, whenever the vectors $\pm\mathbf{r}_j, j = 1 \ldots N$, are of unit length and their tips form a regular polygon in $\mathbb{R}^2$ (for a suitable choice of signs). The bound (73) takes the value *zero* if the coefficient vectors satisfy $\mathbf{a} = \lambda \mathbf{b}$, where $\lambda \in \mathbb{R}$. Consequently, all N observables will be scalar multiples of each other and hence commute, corresponding to arrangements of *minimal* incompatibility.

Furthermore, we conjectured entropic inequalities to hold for more than two continuous variables, analogous in form to the one originally discovered by Hirschman. The sum of the Shannon entropies associated with N directions in phase space is expected to achieve its maximum if the angles between any neighboring directions equal $2\pi/N$. We expect that there will be no states violating the conjectured bound (72) which has been derived from evaluating the N Shannon entropies in a Gaussian state. This N-term generalization of Hirschman's inequality fills a gap concerning entropic inequalities for *continuous* variables while in *finite-dimensional* Hilbert spaces numerous investigations of entropic inequalities for multiple variables have been carried out already.

Our results raise a number of questions which we hope to address in future work. Let us begin by pointing out a surprising formal similarity between the result (73) and the inequality for the sum of standard deviations of two spin observables [32]:

$$\Delta A + \Delta B \geq |\mathbf{A} \times \mathbf{B}| , \tag{75}$$

where $\hat{A} = \mathbf{A} \cdot \hat{\boldsymbol{\sigma}}$ and $\hat{B} = \mathbf{B} \cdot \hat{\boldsymbol{\sigma}}$, with unit vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^3$, and $\hat{\boldsymbol{\sigma}} = \left(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \right)^T$ is a vector operator with Pauli matrices as components. Here, the vectors $\mathbf{A}$ and $\mathbf{B}$ collect coefficients of *different* observables, hence should be compared to the vectors $\mathbf{r}_j$, $j = 1 \ldots N$, and not to the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Is there a simple generalization of (75) valid for the sum of the standard deviations of more than two spin observables? Since the observables $\hat{A}, \hat{B}, \ldots$ will be in a one-to-one-correspondence with N points inside of the unit sphere, a natural bound on the incompatibility of N observables is likely to define a geometric structure in $\mathbb{R}^3$, just as regular polygons in $\mathbb{R}^2$ emerge in the case of N continuous variables.

To conclude, we discuss our results from a fundamental perspective. Heisenberg's uncertainty relation has often been understood to say that one cannot simultaneously associate definite values to both position and momentum of a quantum particle. Kochen-Specker-type arguments [33] formalize this insight by showing that non-contextual value-assignments are algebraically – i.e. not statistically – at odds with quantum predictions. Contradictions arise from dichotomic observables for both discrete [34, 35] and continuous quantum variables [36, 37]. A recent probabilistic approach [38] introduces non-contextual "Kochen-Specker inequalities" which lend themselves to experimental verification. Our results may have implications for similar contextuality arguments given in terms of phase-space translations, along the lines of Refs. [39, 40], for example.

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Appendix A

In this appendix, we derive that the sum inequality (22) also holds for mixed states, with the validity of 18 for pure states being our point of departure. To do so, we first consider the sum of the variances of observables $\hat{A}_1, \hat{A}_2, \ldots$, and derive the inequality

$$\sum_j \Delta_\rho^2 A_j \geq \sum_j \Delta_\psi^2 A_j \,, \tag{76}$$

where $\hat{\rho}$ is any mixed state and $|\psi\rangle$ is a pure state which will depend on $\hat{\rho}$. This relation implies that it is sufficient to consider pure states only when searching for universal bounds on sums of variances.

Let $\hat{A}$ be a self-adjoint operator and suppose that the mixed state $\hat{\rho} = \lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2$ is a convex combination of two density matrices $\hat{\rho}_1$ and $\hat{\rho}_2$, with $\lambda \in [0,1]$. Then, the variance of $\hat{A}$ in the mixture $\hat{\rho}$ is bounded from below by the sum of the variances of the in the states $\hat{\rho}_1$ and $\hat{\rho}_2$, i.e.

$$\Delta^2_\rho A \geq \lambda \Delta^2_{\rho_1} A + (1-\lambda)\Delta^2_{\rho_2} A\,, \tag{77}$$

as follows from the concavity of the variance.

To prove that the variance $\Delta^2_\rho A = \langle A^2 \rangle_\rho - \langle A \rangle^2_\rho$, with $\langle \hat{A} \rangle_\rho \equiv \mathrm{Tr}\left(\hat{A}\hat{\rho}\right)$ etc., *is* concave, we note that

$$\langle \hat{A}^2 \rangle_\rho = \lambda \langle \hat{A}^2 \rangle_{\rho_1} + (1-\lambda)\langle \hat{A}^2 \rangle_{\rho_2}\,, \tag{78}$$

and

$$\langle \hat{A} \rangle^2_\rho = \left(\lambda \langle \hat{A} \rangle_{\rho_1} + (1-\lambda)\langle \hat{A} \rangle_{\rho_2}\right)^2 \leq \lambda \langle \hat{A} \rangle^2_{\rho_1} + (1-\lambda)\langle \hat{A} \rangle^2_{\rho_2}\,, \tag{79}$$

using the convexity of the function $f(x) = x^2$. The inequalities (78) and (79) immediately imply inequality (77). Since one of the variances on the right-hand-side of (77), say $\Delta^2_{\rho_1} A$, must be less than or equal to the left-hand-side, we obtain

$$\Delta^2_\rho A \geq \Delta^2_{\rho_1} A\,. \tag{80}$$

This argument can be extended to a sum of the variances of N Hermitean operators $\hat{A}_j, j = 1\ldots N$, resulting in

$$\sum_j \Delta^2_\rho A_j \geq \sum_j \Delta^2_{\rho_1} A_j\,. \tag{81}$$

To see this, note that the sum of two concave functions (such as $\Delta^2_\rho A$ and $\Delta^2_\rho B$) is also concave which leads to

$$\Delta^2_\rho A + \Delta^2_\rho B \geq \lambda \left(\Delta^2_{\rho_1} A + \Delta^2_{\rho_1} B\right) + (1-\lambda)\left(\Delta^2_{\rho_2} A + \Delta^2_{\rho_2} B\right)\,, \tag{82}$$

for any mixture $\hat{\rho} = \lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2$. Again, one of the two terms in brackets on the right-hand-side of (82) is less than or at most equal to the left-hand-side. Thus, we have shown that the inequality (81) holds for two operators. It is straightforward to include more operators.

Finally, to complete the proof of (76), we need to consider a state with a convex decomposition given by to $\hat{\rho} = \sum_k r_k \hat{P}_k$, where the operators $\hat{P}_k = |\psi_k\rangle\langle\psi_k|$, $k = 1, 2, \ldots$, project onto pure states $|\psi_k\rangle$. Then, for the variance of a single observable $\hat{A}$ we have the bound

$$\Delta^2_\rho A \geq \sum_k r_k \Delta^2_{\psi_k} A \geq \Delta^2_\psi A\,, \tag{83}$$

where $|\psi\rangle$ is one of the states $|\psi_1\rangle, |\psi_2\rangle, \ldots$, for which the variances of the right-hand-side falls below or is equal to the left-hand-side. Since this argument also applies to the sum of variances of observables $\hat{A}_1, \hat{A}_2, \ldots$, the inequality (76) does indeed hold.

Appendix B

In this Appendix, we will show that the product $\mathbf{a} \wedge \mathbf{b}$ is invariant (i) under any phase-space rotation of the position and momentum observables and (ii) under squeezing transformations.

(i) Consider any rotation of position $\hat{q}$ and momentum $\hat{p}$ in phase space,

$$\begin{aligned}\hat{p}_\vartheta &= \hat{p}\cos\vartheta + \hat{q}\sin\vartheta\,,\\ \hat{q}_\vartheta &= -\hat{p}\sin\vartheta + \hat{q}\cos\vartheta\,, \qquad \vartheta\in[0,2\pi]\,.\end{aligned} \tag{84}$$

This commutator-preserving transformation is generated by the unitary

$$\hat{R}_\vartheta = \exp\left[-i\vartheta\left(\hat{p}^2+\hat{q}^2\right)/2\hbar\right]\,, \tag{85}$$

known as the time-evolution operator of a harmonic oscillator with unit mass and frequency. The relations (84) induce linear transformations in the coefficient space $\mathbb{R}^N$, which you obtain upon replacing the symbols $\hat{p}$ and $\hat{q}$ in Eq. (84) by $\mathbf{a}$ and $\mathbf{b}$, respectively. Therefore, the exterior product of the transformed vectors reads

$$\mathbf{a}_\vartheta\wedge\mathbf{b}_\vartheta = (\mathbf{a}\cos\vartheta+\mathbf{b}\sin\vartheta)\wedge(-\mathbf{a}\sin\vartheta+\mathbf{b}\cos\vartheta) = \mathbf{a}\wedge\mathbf{b}\,, \tag{86}$$

confirming the expected invariance.

(ii) Rescaling the observables $\hat{p}$ and $\hat{q}$ is achieved by the unitary operator $\hat{U}=\hat{S}_\gamma$ (see Eq. (28)) which *squeezes* the momentum and position operators according to

$$\begin{aligned}\hat{p}_\gamma &= \gamma\hat{p}\,,\\ \hat{q}_\gamma &= \frac{1}{\gamma}\hat{q}\,, \qquad \gamma\neq 0\,.\end{aligned} \tag{87}$$

It is easy to see that the coefficient vectors in $\mathbb{R}^N$ transform in a covariant way, namely

$$\begin{aligned}\mathbf{a}_\gamma &= \gamma\mathbf{a}\,,\\ \mathbf{b}_\gamma &= \frac{1}{\gamma}\mathbf{b}\,, \qquad \gamma\neq 0\,,\end{aligned} \tag{88}$$

which implies the invariance of the product, $\mathbf{a}_\gamma\wedge\mathbf{b}_\gamma=\mathbf{a}\wedge\mathbf{b}$. Choosing the value

$$\gamma_0 = \left(\frac{|\mathbf{b}|}{|\mathbf{a}|}\right)^{1/2}\,, \tag{89}$$

allows us to introduce new coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ with equal length given by$(|\mathbf{a}|\,|\mathbf{b}|)^{1/2}\equiv |\mathbf{a}\wedge\mathbf{b}|^{1/2}$.